QUESTÃO 06:

Use substituição hiperbólica para resolver a integral $\int \sqrt{1+x^2}\ dx$.

Solução:

Se fizermos $x = senh(u)$ então $dx = \cosh(u)\ du$. Assim,

$$\begin{aligned}
\int \sqrt{1+x^2}\ dx &= \int \sqrt{1+senh^2 u}\ \cosh(u)\ du \\
&= \int \sqrt{\cos h^2 u}\ \cosh(u)\ du \\
&= \int \cos hu \cdot \cosh(u)\ du, \ \text{pois } \cosh(u) = \frac{e^u + e^{-u}}{2} > 0\ \forall\ u \in \mathbb{R} \\
&= \int \cos h^2 u\ du \\
&= \int \left(\frac{e^u + e^{-u}}{2}\right)^2\ du \\
&= \int \left(\frac{e^u + e^{-u}}{2}\right)^2\ du \\
&= \frac{1}{4}\int \left(e^{2u} + 2 + e^{-2u}\right)\ du \\
&= \frac{1}{4}\left(\frac{e^{2u}}{2} + 2u - \frac{e^{-2u}}{2}\right) + C \\
&= \frac{1}{4}\left(\frac{e^{2u} - e^{-2u}}{2}\right) + \frac{u}{2} + C \\
&= \frac{1}{4} senh(2u) + \frac{u}{2} + C, \ \text{pois } senh(u) = \frac{e^u - e^{-u}}{2} \\
&= \frac{1}{4} 2senh(u)\cosh(u) + \frac{u}{2} + C \\
&= \frac{1}{2} senh(u)\cosh(u) + \frac{u}{2} + C.
\end{aligned}$$

Logo, $\int \sqrt{1+x^2}\ dx = \frac{1}{2}\left(x\sqrt{1+x^2} + arcsenh(x)\right) + C.$